Exemple 4

a) M1 $\vec{AB}.\vec{AC} = 0$ car les vecteurs sont orthogonaux

M2 $\vec{AB}.\vec{AC} = \begin{pmatrix} -4 \\ 0 \end{pmatrix} . \begin{pmatrix} 0 \\ 5 \end{pmatrix} = (-4)\times 0 + 0\times 5 = 0$

M1 $\vec{ED}.\vec{EG} = -ED\times EG = -3\times 5 = -15$

M2 $\vec{ED}.\vec{EG} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ -5 \end{pmatrix} = 0\times 0 + 3\times(-5) = -15.$

M1 $\vec{HJ}.\vec{HI} = \vec{HI'}.\vec{HJ} = 4\times 6 = 24$

M2 $\vec{HJ}.\vec{HI} = \begin{pmatrix} 6 \\ 0 \end{pmatrix} . \begin{pmatrix} 4 \\ 2 \end{pmatrix} = 6\times 4 + 0\times 2 = 24$

M1 $\vec{NO}.\vec{QS} = \vec{NO}.\vec{QS'} = -NO\times QS' = -3\times 2 = -6$

M2 $\vec{NO}.\vec{QS} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} . \begin{pmatrix} 2 \\ 4 \end{pmatrix} = -3\times 2 + 0\times 4 = -6.$

M1 $\vec{ZW}.\vec{TV} = 0$ car ces 2 vecteurs sont orthogonaux

M2 $\vec{ZW}.\vec{TV} = \begin{pmatrix} 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 2 \times 3 - 2 \times 3 = 0$

M1 $\vec{VT}.\vec{NO} = \vec{VT}.\vec{TP} = \vec{V'T}.\vec{TP}$

$= V'T \times TP = 3 \times 3 = 9$

M2 $\vec{VT}.\vec{NO} = \begin{pmatrix} -3 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 0 \end{pmatrix} = (-3) \times (-3) + (-3) \times 0$

$= 9$

$\vec{NP}.\vec{KM} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = 1 \times 2 - 4 \times 3 = -10$

M1 $\vec{SQ}.\vec{QR} = \vec{S'Q}.\vec{QR} = -QS' \times QR = -2 \times 4 = -8$

M2 $\vec{SQ}.\vec{QR} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \end{pmatrix} = -2 \times 4 + (-4) \times 0 = -8$

Produit scalaire

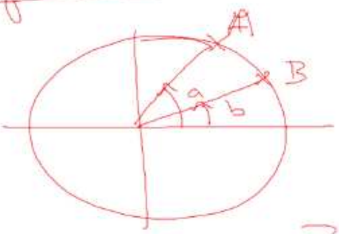
$\vec{AB}\cdot\vec{AC} = AB \times AC \times \cos(\vec{AB}, \vec{AC})$

$\vec{AB}\cdot\vec{AC} = \vec{AB}\cdot\vec{AH} = \begin{cases} AB \times AH & \text{si l'angle est aigu} \\ -AB \times AH & \text{si l'angle est obtus} \end{cases}$

avec H projeté de C sur (AB)

$\vec{AB}\cdot\vec{AC} = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = xx' + yy'$

Propriétés



A a pour coordonnées $(\cos a, \sin a)$

B a pour coordonnées $(\cos b, \sin b)$

$\vec{OA}\cdot\vec{OB} = \begin{pmatrix} \cos a \\ \sin a \end{pmatrix} \cdot \begin{pmatrix} \cos b \\ \sin b \end{pmatrix}$

$= \cos a \cos b + \sin a \sin b$

$\vec{OA}\cdot\vec{OB} = OA \times OB \times \cos(a-b)$

$= 1 \times 1 \times \cos(a-b)$

$= \cos(a-b)$

Donc $\boxed{\cos(a-b) = \cos a \cos b + \sin a \sin b}$

$\cos(a+b) = \cos(a-(-b))$

$$= \cos a \cos(-b) + \sin a \sin(-b)$$

$$\boxed{\cos(a+b) = \cos a \cos b - \sin a \sin b}$$

$$\sin(a+b) = \cos\left(\frac{\pi}{2} - (a+b)\right) = \cos\left(\left(\frac{\pi}{2} - a\right) - b\right)$$

$$= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b$$

$$\boxed{\sin(a+b) = \sin a \cos b + \cos a \sin b}$$

$$\sin(a-b) = \sin(a+(-b))$$

$$= \sin a \cos(-b) + \cos a \sin(-b)$$

$$\boxed{\sin(a-b) = \sin a \cos b - \cos a \sin b}$$

Exemple 4 bis

a) Déterminer $\cos\frac{\pi}{12}$ et $\sin\frac{\pi}{12}$

$$\cos\left(\frac{\pi}{12}\right) = \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{1}{2}\times\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{2}+\sqrt{6}}{4}$$

$$\sin\frac{\pi}{12} = \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}$$

$$= \frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2} - \frac{1}{2}\times\frac{\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

b) Déterminer $\cos\frac{\pi}{8}$ et $\sin\frac{\pi}{8}$

$$\cos\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{8}+\frac{\pi}{8}\right) = \cos\frac{\pi}{8}\cos\frac{\pi}{8} - \sin\frac{\pi}{8}\sin\frac{\pi}{8}$$

$$= \cos^2\frac{\pi}{8} - \sin^2\frac{\pi}{8} = \cos^2\frac{\pi}{8} - \left(1-\cos^2\frac{\pi}{8}\right)$$

$$= 2\cos^2\frac{\pi}{8} - 1$$

Donc $\frac{\sqrt{2}}{2} = 2\cos^2\frac{\pi}{8} - 1 \iff \cos^2\frac{\pi}{8} = \dfrac{\frac{\sqrt{2}}{2}+1}{2}$

$$\iff \cos^2\frac{\pi}{8} = \frac{\sqrt{2}+2}{4}$$

Comme $\frac{\pi}{8} \in \left[0;\frac{\pi}{2}\right]$, $\cos\frac{\pi}{8} \geq 0$

donc $\cos\frac{\pi}{8} = \sqrt{\frac{\sqrt{2}+2}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$

$$\sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{8}+\frac{\pi}{8}\right) = \sin\frac{\pi}{8}\cos\frac{\pi}{8} + \cos\frac{\pi}{8}\sin\frac{\pi}{8}$$

$$= 2\sin\frac{\pi}{8}\cos\frac{\pi}{8}$$

donc $\frac{\sqrt{2}}{2} = 2\sin\frac{\pi}{8}\cos\frac{\pi}{8} \iff \frac{\sqrt{2}}{2} = 2\,\frac{\sqrt{2+\sqrt{2}}}{2}\sin\frac{\pi}{8}$

$$\iff \sin\frac{\pi}{8} = \frac{\sqrt{2}}{2\sqrt{2+\sqrt{2}}}$$

$$\iff \sin\frac{\pi}{8} = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2\sqrt{2+\sqrt{2}}\sqrt{2-\sqrt{2}}}$$

$$\iff \sin\frac{\pi}{8} = \frac{\sqrt{2}\sqrt{2-\sqrt{2}}}{2\sqrt{2}}$$

$$\iff \boxed{\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}}$$

Exemple 5

$$\overrightarrow{AB}.\overrightarrow{CD}+\overrightarrow{AC}.\overrightarrow{DB}+\overrightarrow{AD}.\overrightarrow{BC}$$

$$=\overrightarrow{AB}.\overrightarrow{CD}+(\overrightarrow{AB}+\overrightarrow{BC}).\overrightarrow{DB}+(\overrightarrow{AB}+\overrightarrow{BD}).\overrightarrow{BC}$$

$$=\underline{\overrightarrow{AB}.}\overrightarrow{CD}+\underline{\overrightarrow{AB}.}\overrightarrow{DB}+\overrightarrow{BC}.\overrightarrow{DB}+\underline{\overrightarrow{AB}.}\overrightarrow{BC}+\overrightarrow{BD}.\overrightarrow{BC}$$

$$=\overrightarrow{AB}.(\overrightarrow{CD}+\overrightarrow{DB}+\overrightarrow{BC})+\overrightarrow{BC}.(\overrightarrow{DB}+\overrightarrow{BD})$$

$$=\overrightarrow{AB}.\vec{0}+\overrightarrow{BC}.\vec{0}=0+0=0.$$

Exemple 6

1) $\vec{u}.\vec{v}=0 \iff \begin{pmatrix}1\\3\end{pmatrix}.\begin{pmatrix}2\\m\end{pmatrix}=0$

$\iff 1\times 2+3\times m=0$

$\iff 2+3m=0$

$\iff m=-\frac{2}{3}.$

3) $\vec{u}.\vec{v}=0 \iff \begin{pmatrix}2m+1\\1\end{pmatrix}.\begin{pmatrix}m+1\\-3\end{pmatrix}=0$

$\iff (2m+1)(m+1)-3=0$

$\iff 2m^2+3m-2=0$

$\iff (m+2)(2m-1)=0$

$\iff m=-2$ ou $m=\frac{1}{2}.$

Propriété 10

$$MA^2 + MB^2 = \vec{MA}^2 + \vec{MB}^2$$
$$= (\vec{MI} + \vec{IA})^2 + (\vec{MI} + \vec{IB})^2$$
$$= \vec{MI}^2 + 2\vec{MI}.\vec{IA} + \vec{IA}^2 + \vec{MI}^2 + 2\vec{MI}.\vec{IB} + \vec{IB}^2$$
$$= 2\vec{MI}^2 + 2\vec{MI}.(\vec{IA} + \vec{IB}) + \vec{IA}^2 + \vec{IB}^2$$
$$= 2MI^2 + 2\vec{MI}.\vec{0} + IA^2 + IB^2$$
$$= 2MI^2 + 0 + \frac{1}{4}AB^2 + \frac{1}{4}AB^2$$
$$= 2MI^2 + \frac{1}{2}AB^2$$

Exemple 7

$AB = 2 \quad AC = 3 \quad BC = 4$

$$AB^2 = AC^2 + BC^2 - 2AC \times BC \times \cos\widehat{ACB}$$
$$\Longleftrightarrow 4 = 9 + 16 - 24\cos\widehat{ACB}$$
$$\Longleftrightarrow \cos\widehat{ACB} = \frac{21}{24}$$
$$\Longleftrightarrow \cos\widehat{ACB} = \frac{7}{8}$$

Ainsi $\widehat{ACB} \simeq 28{,}9°$ à 10^{-1} près