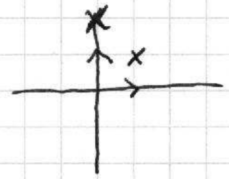
TD n° 15 Nombres complexes (2)

Exercice 4

1/ Si on place les images de $1+i$ et $2i$



on voit que : $1+i = [\sqrt{2}; \frac{\pi}{4}]$

et $2i = [2; \frac{\pi}{2}]$.

Ainsi $\arg(z_1 \times z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{4} + \frac{\pi}{2} = \frac{3\pi}{4}$

$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$.

2/ $|z_1| = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$.

Soit θ un argument de z_1

$$\left.\begin{array}{l} \cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = \frac{1}{2} \end{array}\right\} \text{donc } \theta = \frac{\pi}{6} \text{ convient.}$$

$|z_2| = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$

Soit θ' un argument de z_2

$$\left.\begin{array}{l} \cos\theta' = \frac{1}{2} \\ \sin\theta' = \frac{\sqrt{3}}{2} \end{array}\right\} \text{donc } \theta' = \frac{\pi}{3} \text{ convient.}$$

Ainsi $\arg(z_1 \times z_2) = \arg(z_1) + \arg(z_2) = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$.

$\arg\left(\frac{z_1^2}{z_2}\right) = \arg(z_1^2) - \arg(z_2) = 2\arg(z_1) - \arg(z_2)$

$= 2 \times \frac{\pi}{6} - \frac{\pi}{3} = \frac{\pi}{3} - \frac{\pi}{3} = 0$.

Exercice 5

a) Graphiquement on voit que : $1+i = [\sqrt{2}; \frac{\pi}{4}]$

Donc $z_1 = [\sqrt{2}; \frac{\pi}{4}]^5 = [\sqrt{2}^5; \frac{5\pi}{4}] = [4\sqrt{2}; \frac{5\pi}{4}]$

D'où $z_1 = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 4\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\left(-\frac{\sqrt{2}}{2}\right)\right)$

$= -4 - 4i$

b) Graphiquement on voit que : $1-i = [\sqrt{2}; -\frac{\pi}{4}]$

Donc $z_2 = [\sqrt{2}; -\frac{\pi}{4}]^6 = [\sqrt{2}^6; -\frac{6\pi}{4}] = [8; -\frac{3\pi}{2}] = [8; \frac{\pi}{2}]$

D'où $z_2 = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 8i$

c) $|1+i\sqrt{3}| = \sqrt{1+\sqrt{3}^2} = \sqrt{4} = 2$

Soit θ un argument de $1+i\sqrt{3}$

$\left.\begin{array}{l}\cos\theta = \frac{1}{2}\\ \sin\theta = \frac{\sqrt{3}}{2}\end{array}\right\}$ d'où $\theta = \frac{\pi}{3}$ convient. Donc $1+i\sqrt{3} = [2;\frac{\pi}{3}]$

Ainsi $z_3 = [2;\frac{\pi}{3}]^7 = [2^7;\frac{7\pi}{3}] = [128;\frac{\pi}{3}] = 128(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$z_3 = 128(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 64 + 64i\sqrt{3}$

d) Graphiquement on voit que $(-1+i) = [\sqrt{2};\frac{3\pi}{4}]$

Donc $z_4 = [\sqrt{2};\frac{3\pi}{4}]^4 = [\sqrt{2}^4; 4\times\frac{3\pi}{4}] = [4;3\pi] = [4;\pi]$

$z_4 = 4(\cos\pi + i\sin\pi) = 4\times(-1) = -4$

e) $|2-2i\sqrt{3}| = \sqrt{2^2+(-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$

Soit θ un argument de $2-2i\sqrt{3}$

$\left.\begin{array}{l}\cos\theta = \frac{2}{4} = \frac{1}{2}\\ \sin\theta = -\frac{2\sqrt{3}}{4} = -\frac{\sqrt{3}}{2}\end{array}\right\}$ donc $\theta = -\frac{\pi}{3}$ convient.

Ainsi $z_5 = [4;-\frac{\pi}{3}]^5 = [4^5;-\frac{5\pi}{3}] = [1024;\frac{\pi}{3}]$

$= 1024(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1024(\frac{1}{2} + i\frac{\sqrt{3}}{2})$

$= 512 + 512i\sqrt{3}$

f) $|\sqrt{3}+i| = \sqrt{\sqrt{3}^2+1^2} = 2$

Soit θ un argument de $\sqrt{3}+i$

$\left.\begin{array}{l}\cos\theta = \frac{\sqrt{3}}{2}\\ \sin\theta = \frac{1}{2}\end{array}\right\}$ donc $\theta = \frac{\pi}{6}$ convient. Ainsi $\sqrt{3}+i = [2;\frac{\pi}{6}]$

$\sqrt{3}-i = \overline{\sqrt{3}+i} = \overline{[2;\frac{\pi}{6}]} = [2;-\frac{\pi}{6}]$

Ainsi $z_6 = \frac{[2;\frac{\pi}{6}]^9}{[2;-\frac{\pi}{6}]^{11}} = \frac{[2^9;\frac{9\pi}{6}]}{[2^{11};-\frac{11\pi}{6}]} = [\frac{2^9}{2^{11}};\frac{9\pi}{6}+\frac{11\pi}{6}]$

$= [\frac{1}{2^2};\frac{20\pi}{6}] = [\frac{1}{4};\frac{4\pi}{3}] = \frac{1}{4}(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$

$= \frac{1}{4}(\frac{-1}{2} + i(-\frac{\sqrt{3}}{2})) = -\frac{1}{8} - \frac{i\sqrt{3}}{8}$

g) $\left|\frac{1+i\sqrt{3}}{2}\right| = \sqrt{(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

Soit θ un argument de $\frac{1+i\sqrt{3}}{2}$

$$\left.\begin{array}{l} \cos\theta = \frac{\frac{1}{2}}{1} = \frac{1}{2} \\ \sin\theta = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \end{array}\right\} \text{ donc } \theta = \frac{\pi}{3} \text{ convient.}$$

Ainsi $z_f = \left[1;\frac{\pi}{3}\right]^5 = \left[1^5;\frac{5\pi}{3}\right] = \left[1;\frac{5\pi}{3}\right] = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$

$= \frac{1}{2} - i\frac{\sqrt{3}}{2}$

g) $\sqrt{2}+i\sqrt{2} = \sqrt{2}(1+i)$ et $\sqrt{2}-i\sqrt{2} = \sqrt{2}(1-i)$

Graphiquement on a : $1+i = \left[\sqrt{2};\frac{\pi}{4}\right]$ et $1-i = \left[\sqrt{2};-\frac{\pi}{4}\right]$

Ainsi $\sqrt{2}+i\sqrt{2} = \sqrt{2}(1+i) = [\sqrt{2};0]\left[\sqrt{2};\frac{\pi}{4}\right] = \left[\sqrt{2}^2;\frac{\pi}{4}\right] = \left[2;\frac{\pi}{4}\right]$

$\sqrt{2}-i\sqrt{2} = \sqrt{2}(1-i) = [\sqrt{2};0]\left[\sqrt{2};-\frac{\pi}{4}\right] = \left[\sqrt{2}^2;-\frac{\pi}{4}\right] = \left[2;-\frac{\pi}{4}\right]$

Donc $z_g = \frac{[2;\pi/4]^3}{\left[2;-\frac{\pi}{4}\right]^5} = \left[\frac{2^3}{2^5};\frac{3\pi}{4}+\frac{5\pi}{4}\right] = \left[\frac{1}{4};2\pi\right] = \left[\frac{1}{4};0\right]$

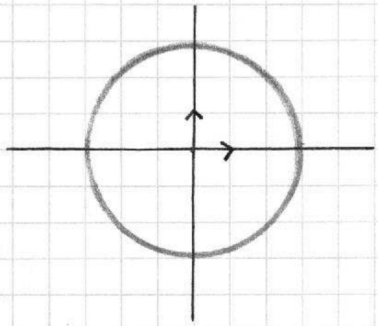
$= \frac{1}{4}(\cos 0 + i\sin 0) = \frac{1}{4}$

Exercice 6

1] $|z| = 3 \Leftrightarrow |z_M - z_O| = 3 \Leftrightarrow |z_{\vec{OM}}| = 3$

$\Leftrightarrow OM = 3$

L'ensemble cherché est le cercle de centre O et de rayon 3.

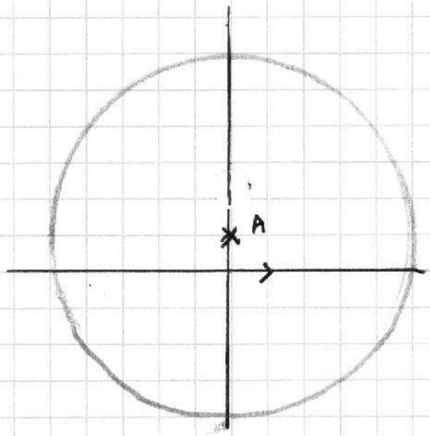


2] Soit A le point d'affixe i

$|z-i| = 5 \Leftrightarrow |z_M - z_A| = 5 \Leftrightarrow |z_{\vec{AM}}| = 5$

$\Leftrightarrow AM = 5$

L'ensemble cherché est le cercle de centre A et de rayon 5.

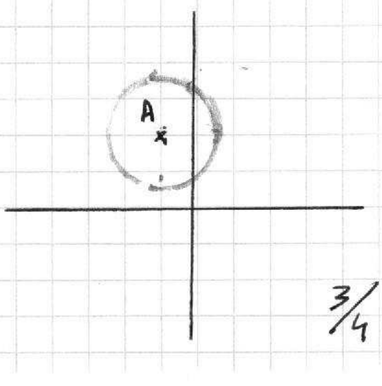


3] Soit A le point d'affixe $-1+2i$

$|z+1-2i| = \sqrt{2} \Leftrightarrow |z_M - z_A| = \sqrt{2} \Leftrightarrow |z_{\vec{AM}}| = \sqrt{2}$

$\Leftrightarrow AM = \sqrt{2}$

L'ensemble cherché est le cercle de A et de rayon $\sqrt{2}$



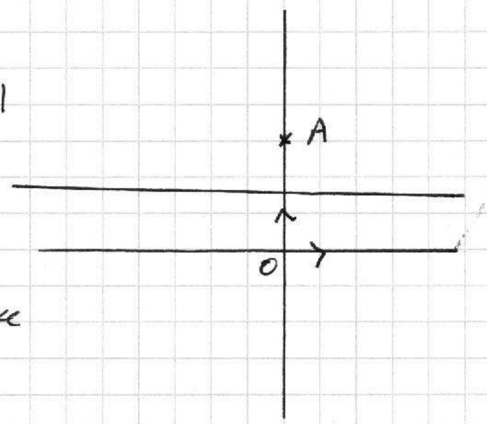
4] Soit A le point d'affixe $3i$

$|z| = |z-3i| \iff |z_M - z_O| = |z_M - z_A|$

$\iff |z_{\vec{OM}}| = |z_{\vec{AM}}|$

$\iff OM = AM$

L'ensemble cherché est la médiatrice de $[OA]$.



5] Soient A et B les points d'affixes respectives $1-2i$ et 1

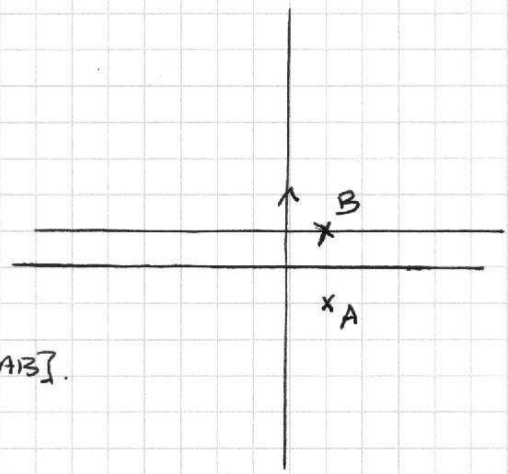
$|z-1+2i| = |1-z| \iff |z_M - z_A| = |z_B - z_M|$

$\iff |z_{\vec{AM}}| = |z_{\vec{MB}}|$

$\iff AM = MB$

$\iff AM = BM$

L'ensemble cherché est la médiatrice de $[AB]$.



6] $|1+3i| = \sqrt{1+3^2} = \sqrt{10}$

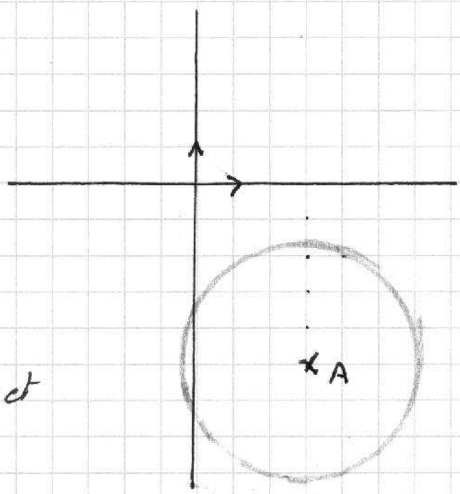
soit A le point d'affixe $3-5i$

$|z-3+5i| = |1+3i| \iff |z_M - z_A| = \sqrt{10}$

$\iff |z_{\vec{AM}}| = \sqrt{10}$

$\iff AM = \sqrt{10}$

L'ensemble cherché est le cercle de centre A et de rayon $\sqrt{10}$



7] $|\bar{z} - i| = |iz + 1 - 3i| \iff |\bar{z} + \bar{i}| = |i(z - i - 3)|$

$\iff |\overline{z+i}| = |i| \times |z - i - 3|$

$\iff |z+i| = 1 \times |z-i-3|$

Soient A et B les points d'affixes $-i$ et $3+i$

Donc $|\bar{z} - i| = |iz+1-3i| \iff |z_M - z_A| = |z_M - z_B|$

$\iff |z_{\vec{AM}}| = |z_{\vec{BM}}| \iff AM = BM$

L'ensemble cherché est la médiatrice de $[AB]$