TD n° 15 : Nombres complexes (2)

Exercice 7

1) $z_2 = \frac{\sqrt{3}+i}{4} - i = \frac{\sqrt{3}}{4} - \frac{3}{4}i$

$z_3 = \bar{z}_1 - i = \frac{\sqrt{3}-i}{4} - i = \frac{\sqrt{3}}{4} - \frac{5}{4}i$

$|z_3 - z_1| = \left|\frac{\sqrt{3}}{4} - \frac{5}{4}i - \frac{\sqrt{3}+i}{4}\right| = \left|\frac{\sqrt{3}}{4} - \frac{5}{4}i - \frac{\sqrt{3}}{4} - \frac{i}{4}\right| = \left|-\frac{3}{2}i\right| = \frac{3}{2}$

$|z_3 - z_2| + |z_2 - z_1| = \left|\frac{\sqrt{3}}{4} - \frac{5}{4}i - \left(\frac{\sqrt{3}}{4} - \frac{3}{4}i\right)\right| + \left|\frac{\sqrt{3}}{4} - \frac{3}{4}i - \left(\frac{\sqrt{3}+i}{4}\right)\right|$

$= \left|\frac{\sqrt{3}}{4} - \frac{5}{4}i - \frac{\sqrt{3}}{4} + \frac{3}{4}i\right| + \left|\frac{\sqrt{3}}{4} - \frac{3}{4}i - \frac{\sqrt{3}}{4} - \frac{i}{4}\right|$

$= \left|-\frac{1}{2}i\right| + |-i|$

$= \frac{1}{2} + 1 = \frac{3}{2}$.

Ainsi $|z_3 - z_1| = |z_3 - z_2| + |z_2 - z_1| \Leftrightarrow M_1M_3 = M_2M_3 + M_1M_2$

cette dernière égalité parce que les points M_1, M_2 et M_3 sont alignés et que M_2 appartient à $[M_1M_3]$.

2) $z_1 = \frac{1}{4}(\sqrt{3}+i)$

$|\sqrt{3}+i| = \sqrt{4} = 2$

Soit θ un argument de $\sqrt{3}+i$

$\left.\begin{array}{l}\cos\theta = \frac{\sqrt{3}}{2}\\ \sin\theta = \frac{1}{2}\end{array}\right\}$ donc $\theta = \frac{\pi}{6}$ convient.

donc $\arg(z_1) = \arg\left(\frac{1}{4}\right) + \arg(\sqrt{3}+i)$

$= 0 + \frac{\pi}{6} = \frac{\pi}{6}$.

$z_2 = \frac{\sqrt{3}}{4} - \frac{3}{4}i$

$|z_2| = \sqrt{\frac{3}{16} + \frac{9}{16}} = \frac{\sqrt{12}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$.

Soit θ un argument de z_2.

$\left.\begin{array}{l}\cos\theta = \dfrac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2}} = \frac{1}{2}\\ \sin\theta = \dfrac{-\frac{3}{4}}{\frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}}{2}\end{array}\right\}$ donc $\theta = -\frac{\pi}{3}$ convient.

Ainsi $\arg(z_2) = -\frac{\pi}{3}$.

$(\overrightarrow{OM_1}, \overrightarrow{OM_3}) = (\overrightarrow{OM_1}, \vec{u}) + (\vec{u}, \overrightarrow{OM_3}) = (\overrightarrow{OM_1}, \vec{u}) - (\overrightarrow{OM_3}, \vec{u})$

$= \arg z_{\overrightarrow{OM_1}} - \arg z_{\overrightarrow{OM_3}} = \arg z_1 - \arg z_3 = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$

Donc le triangle OM_1M_3 est rectangle en O.

Exercice 9

1] a) $|z| = \sqrt{1+3} = 2$

Soit θ un argument de z

$\left.\begin{array}{l}\cos\theta = \frac{1}{2} \\ \sin\theta = -\frac{\sqrt{3}}{2}\end{array}\right\}$ donc $\theta = -\frac{\pi}{3}$ convient.

Ainsi $z = 2e^{-i\frac{\pi}{3}}$

b) $z^2 = (2e^{-i\frac{\pi}{3}})^2 = 4e^{-2i\frac{\pi}{3}}$

$z^3 = (2e^{-i\frac{\pi}{3}})^3 = 8e^{-3i\frac{\pi}{3}} = 8e^{-i\pi} = 8e^{i\pi}$

$z^{22} = (2e^{-i\frac{\pi}{3}})^{22} = 2^{22}e^{-22i\frac{\pi}{3}} = 2^{22}e^{-22i\frac{\pi}{3}+24i\frac{\pi}{3}} = 2^{22}e^{2i\frac{\pi}{3}}$

2] a) $|z| = |-2+2i| = \sqrt{4+4} = 2\sqrt{2}$

Soit θ un argument de z

$\left.\begin{array}{l}\cos\theta = \frac{-2}{2\sqrt{2}} = \frac{-1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \\ \sin\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}\end{array}\right\}$ donc $\theta = \frac{3\pi}{4}$ convient.

Ainsi $z = 2\sqrt{2}e^{3i\frac{\pi}{4}}$

b) $(-z)^4 = z^4 = (2\sqrt{2}\,e^{3i\frac{\pi}{4}})^4 = (2\sqrt{2})^4 e^{3i\pi} = 64e^{i\pi}$

$\frac{1}{z^3} = \frac{1}{(2\sqrt{2}e^{3i\frac{\pi}{4}})^3} = \frac{1}{(2\sqrt{2})^3} \times e^{-9i\frac{\pi}{4}} = \frac{1}{16\sqrt{2}}e^{-i\frac{\pi}{4}}$

Exercice 10

$z = \frac{(9\sqrt{3}+3i)(2-i\sqrt{3})}{(2+i\sqrt{3})(2-i\sqrt{3})} = \frac{18\sqrt{3}-27i+6i+3\sqrt{3}}{4+3} = \frac{21\sqrt{3}-21i}{7}$

$z = 3(\sqrt{3}-i)$

$|\sqrt{3}-i| = \sqrt{3+1} = 2.$

Soit θ un argument de $\sqrt{3}+i$

$\left.\begin{array}{l}\cos\theta = \frac{\sqrt{3}}{2} \\ \sin\theta = \frac{-1}{2}\end{array}\right\}$ donc $\theta = -\frac{\pi}{6}$ convient.

Ainsi $z = 3\times 2e^{-i\frac{\pi}{6}} = 6e^{-i\frac{\pi}{6}}$

Exercice 11

1) OACB parallélogramme $\iff \vec{OA} = \vec{BC}$

$$\iff z_{\vec{OA}} = z_{\vec{BC}}$$

$$\iff z_A - z_O = z_C - z_B$$

$$\iff 2e^{i\frac{\pi}{3}} - 0 = c - e^{-i\frac{\pi}{6}}$$

$$\iff c = 2e^{i\frac{\pi}{3}} + e^{i\pi/6}$$

$$\iff c = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) + \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$$

$$\iff c = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\iff c = 1 + i\sqrt{3} + \frac{\sqrt{3}}{2} + \frac{1}{2}i$$

$$\iff c = \frac{2+\sqrt{3}}{2} + \frac{1+2\sqrt{3}}{2}i$$

2) $(\vec{OA}, \vec{OB}) = (\vec{OA}, \vec{u}) + (\vec{u}, \vec{OB}) = -(\vec{u}, \vec{OA}) + (\vec{u}, \vec{OB})$

$= -\arg z_{\vec{OA}} + \arg z_{\vec{OB}} = -\arg a + \arg b$

$= -\frac{\pi}{3} - \frac{\pi}{6} = -\frac{\pi}{2}.$

3) L'angle $\widehat{AOB}$ est droit.

Le parallélogramme OACB est un rectangle.