TD n° 16 : Intégration

Exercice 7

1) $\int_0^2 t\,dt = \left[\frac{t^2}{2}\right]_0^2 = \frac{4}{2} - 0 = 2$

2) $\int_0^2 x\,dt = \left[xt\right]_0^2 = 2x - 0 = 2x$

3) $\int_0^2 x\,dx = \int_0^2 t\,dt = 2$

4) $\int_1^2 (3x^2 - x + \frac{1}{x})dx = \left[x^3 - \frac{1}{2}x^2 + \ln x\right]_1^2$

$= (2^3 - \frac{1}{2}2^2 + \ln 2) - (1^3 - \frac{1}{2}.1^2 + \ln 1)$

$= 6 + \ln 2 - 1 + \frac{1}{2}$

$= \frac{11}{2} + \ln 2$

5) $\int_{-2}^3 (2t-1)(t^2-t+1)dt = \left[\frac{1}{2}(t^2-t+1)^2\right]_{-2}^3$

$= \frac{1}{2}\left[(3^2-3+1)^2 - ((-2)^2-(-2)+1)^2\right]$

$= \frac{1}{2}(7^2 - 7^2) = 0$

6) $\int_0^{\frac{\pi}{4}} \sin(3x)\,dx = \left[-\frac{1}{3}\cos(3x)\right]_0^{\pi/4} = -\frac{1}{3}\cos\frac{3\pi}{4} + \frac{1}{3}\cos 0$

$= -\frac{1}{3}\left(-\frac{\sqrt{2}}{2}\right) + \frac{1}{3} = \frac{\sqrt{2}+2}{6}$

7) $\int_1^2 \frac{t^3+1}{t^4+4t+1}dt = \frac{1}{4}\int_1^2 \frac{4t^3+4}{t^4+4t+1}dt = \left[\frac{1}{4}\ln(t^4+4t+1)\right]_1^2$

$= \frac{1}{4}\left[\ln(2^4+4\times 2+1) - \ln(1+4+1)\right]$

$= \frac{1}{4}(\ln 25 - \ln 6) = \frac{1}{4}\ln\frac{25}{6}$

8) $\int_{-2}^{-1}\left(1 - \frac{3}{x}\right)dx = \left[x - 3\ln(-x)\right]_{-2}^{-1}$

$= (-1 - 3\ln 1) - (-2 - 3\ln 2)$

$= -1 + 2 + 3\ln 2$

$= 3\ln 2 + 1$

9) $\displaystyle\int_{-1}^{1} e^{3x+4}\,dx = \frac{1}{3}\int_{-1}^{1} 3e^{3x+4}\,dx = \frac{1}{3}\left[e^{3x+4}\right]_{-1}^{1}$

$= \frac{1}{3}\left(e^{7} - e\right)$

10) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{6}} \sin x \cos^3 x\,dx = \left[-\frac{1}{4}\cos^4 x\right]_{-\frac{\pi}{4}}^{\frac{\pi}{6}}$

$= -\frac{1}{4}\left[\left(\cos\frac{\pi}{6}\right)^4 - \cos\left(-\frac{\pi}{4}\right)^4\right]$

$= \frac{-1}{4}\left(\left(\frac{\sqrt{3}}{2}\right)^4 - \left(\frac{\sqrt{2}}{2}\right)^4\right) = \frac{-1}{64}\left(9-4\right)$

$= -\frac{5}{64}$

11) $\displaystyle\int_{e}^{e^2} \frac{1}{x\ln x}\,dx = \int_{e}^{e^2} \frac{\frac{1}{x}}{\ln x}\,dx = \left[\ln(\ln x)\right]_{e}^{e^2}$

$= \ln(\ln e^2) - \ln(\ln e)$

$= \ln 2 - \ln 1 = \ln 2$

12) On effectue une intégration par parties.

On pose $u = t$ $\quad v' = \sin t$

$u' = 1$ $\quad v = -\cos t$

$\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} t\sin t\,dt = \left[-t\cos t\right]_{-\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} -\cos t\,dt$

$= -\frac{\pi}{2}\cos\frac{\pi}{2} - \frac{\pi}{4}\cos\frac{\pi}{4} + \displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \cos t\,dt$

$= 0 - \frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \left[\sin t\right]_{-\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -\frac{\pi\sqrt{2}}{8} + \sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{4}\right)$

$= -\frac{\pi\sqrt{2}}{8} + 1 + \frac{\sqrt{2}}{2}$

$= \frac{8 + 4\sqrt{2} - \pi\sqrt{2}}{8}$

13) On effectue une intégration par parties

On pose : $u = \ln x$ $\quad v' = x$

$u' = \frac{1}{x}$ $\quad v = \frac{1}{2}x^2$

$$\int_1^e x\ln x\,dx = \left[\frac{1}{2}x^2\ln x\right]_1^e - \int_1^e \frac{1}{x}\times\frac{1}{2}x^2\,dx$$

$$= \frac{1}{2}e^2\ln e - \frac{1}{2}1^2\ln 1 - \int_1^e \frac{1}{2}x\,dx$$

$$= \frac{1}{2}e^2 - \left[\frac{x^2}{4}\right]_1^e = \frac{1}{2}e^2 - \left(\frac{e^2}{4} - \frac{1}{4}\right)$$

$$= \frac{e^2+1}{4}$$

14) $$\int_1^4 \frac{x^3+2x^2+4x-1}{x^2}\,dx = \int_1^4 \left(x+2+\frac{4}{x}-\frac{1}{x^2}\right)dx$$

$$= \left[\frac{x^2}{2}+2x+4\ln x+\frac{1}{x}\right]_1^4$$

$$= \left(8+8+4\ln 4+\frac{1}{4}\right) - \left(\frac{1}{2}+2+4\ln 1+1\right)$$

$$= 16+8\ln 2+\frac{1}{4}-\frac{1}{2}-3$$

$$= \frac{51}{4}+8\ln 2$$

15) $$\int_1^e \ln t\,dt + \int_e^1 \ln t\,dt = \int_1^1 \ln t\,dt = 0$$

16) $$\int_1^e \ln t\,dt + \int_1^e \left(t+\ln\frac{1}{t}\right)dt = \int_1^e (\ln t + t - \ln t)\,dt$$

$$= \int_1^e t\,dt$$

$$= \left[\frac{t^2}{2}\right]_1^e$$

$$= \frac{e^2-1}{2}$$

17) $$\int_{-2}^0 x e^{x^2}\,dx = \left[\frac{1}{2}e^{x^2}\right]_{-2}^0 = \frac{1}{2}(e^0-e^4) = \frac{1-e^4}{2}$$

18) On effectue une intégration par parties

On pose : $u = x$ $\quad v' = e^x$

$u' = 1$ $\quad v = e^x$

$$\int_{-1}^0 xe^x\,dx = \left[xe^x\right]_{-1}^0 - \int_{-1}^0 e^x\,dx$$

$$= (0+e^{-1}) - \left[e^x\right]_{-1}^0$$

$$= e^{-1} - 1 + e^{-1}$$

$$= \frac{2}{e} - 1$$

19) On effectue une intégration par parties.

On pose : $u = \ln x \qquad v' = \frac{1}{x^2}$

$u' = \frac{1}{x} \qquad v = -\frac{1}{x}$

$$\int_1^e \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^e - \int_1^e \frac{1}{x}\times\left(\frac{-1}{x}\right)dx$$

$$= \left(-\frac{\ln e}{e} + \frac{\ln 1}{1}\right) - \left[\frac{1}{x}\right]_1^e$$

$$= \frac{-1}{e} - \left(\frac{1}{e} - 1\right)$$

$$= 1 - \frac{2}{e}$$

20) On cherche a et b réels tels que : $\frac{1}{x^2-9} = \frac{a}{x-3} + \frac{b}{x+3}$

$$\frac{1}{x^2-9} = \frac{a}{x-3} + \frac{b}{x+3}$$

$\Leftrightarrow 1 = a(x+3) + b(x-3)$

$\Leftrightarrow 1 = (a+b)x + 3(a-b)$

$\Leftrightarrow \begin{cases} a+b=0 \\ 3(a-b)=1 \end{cases}$

$\Leftrightarrow \begin{cases} a=-b \\ 3\times(-2b)=1 \end{cases}$

$\Leftrightarrow \begin{cases} b=\frac{-1}{6} \\ a=\frac{1}{6} \end{cases}$

Ainsi $\frac{1}{x^2-9} = \frac{1/6}{x-3} - \frac{1/6}{x+3}$

Donc $\int_{-2}^0 \frac{1}{x^2-9}\,dx = \frac{1}{6}\int_{-2}^0 \left(\frac{1}{x-3} - \frac{1}{x+3}\right)dx$

$$= \frac{1}{6}\left[\ln(-x+3) - \ln(x+3)\right]_{-2}^0$$

$$= \frac{1}{6}\left((\ln 3 - \ln 3) - (\ln 5 - \ln 1)\right)$$

$$= \frac{1}{6}(0 - \ln 5)$$

$$= -\frac{\ln 5}{6}$$